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THE SPECTRAL GAP AND PERTURBATION BOUNDS FOR REVERSIBLE CONTINUOUS-TIME MARKOV CHAINS

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Abstract

We show that, for reversible continuous-time Markov chains, the closeness of the nonzero eigenvalues of the generator to zero provides complete information about the sensitivity of the distribution vector to perturbations of the generator. Our results hold for both the transient and the stationary states.

Keywords: Markov chain; reversibility; eigenvalue; spectral gap; condition number; perturbation bound; sensitivity analysis

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1. Introduction

In sensitivity analysis for Markov chains, it is important to study the relationship between the stability under perturbation and other properties of the chain. Eigenvalues of the generator or the transition probability matrix are among the factors which establish such relations. In [3] and [6], it was shown that, for the stationary distribution of a finite discrete-time Markov chain, the stability is controlled by the closeness of the nonunit eigenvalues of the transition probability matrix to unity; see also [2]. In this paper we discuss perturbation bounds which hold for both the transient and the stationary states. We prove that, for reversible chains in continuous time, the sensitivity of a chain under perturbation of the generator is governed primarily by the spectral gap of the generator. We thus demonstrate a fundamental connection between the stability of a reversible chain and its speed of convergence to stationarity, allowing us to simplify the analysis of Markov chain models arising from applications; see [4].

2. Eigenvalue bounds for a condition number

Consider two continuous-time Markov chains, $X(t)$ and $\tilde{X}(t),\ t \geq 0$, with finite state space $\mathcal{S} = \{0, 1, \dots, N\},\ N \geq 1$, and respective generators $\boldsymbol{Q}$ and $\tilde{\boldsymbol{Q}}$ (with $\boldsymbol{Q} \neq \tilde{\boldsymbol{Q}}$). Denote by $\boldsymbol{p}(t)$ and $\tilde{\boldsymbol{p}}(t)$ the distribution vectors of $X(t)$ and $\tilde{X}(t)$:

$$\boldsymbol{p}(t) = \boldsymbol{p}(0)\exp(t\boldsymbol{Q}), \qquad \tilde{\boldsymbol{p}}(t) = \tilde{\boldsymbol{p}}(0)\exp(t\tilde{\boldsymbol{Q}}).$$

In this paper we study perturbations of the generator, so we set $\tilde{\boldsymbol{p}}(0) = \boldsymbol{p}(0)$. We also assume that $X(t)$ has a unique stationary distribution $\boldsymbol{\pi} = (\pi_i)$.

We regard all vectors as row vectors, and $\|\cdot\|$ denotes the l_1-norm (absolute entry sum) for vectors and the corresponding induced norm (maximum absolute row sum) for matrices.

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Define

$$\beta(t) = \tfrac{1}{2} \max_{i,j \in \mathcal{S}} \|(\boldsymbol{e}_i - \boldsymbol{e}_j) \exp(t\boldsymbol{Q})\|,$$

where $\boldsymbol{e}_i$ is the vector whose ith entry is 1 and whose other entries are all 0. Set

$$\tau_1 = \inf\{t > 0 : \beta(t) \le \mathrm{e}^{-1}\}.$$

Theorem 1. *The following inequality holds:*

$$\sup_{t \ge 0} \|\boldsymbol{z}(t)\| < \frac{\mathrm{e}\tau_1}{\mathrm{e} - 1} \|\boldsymbol{E}\|,$$

where $\boldsymbol{z}(t) = \tilde{\boldsymbol{p}}(t) - \boldsymbol{p}(t)$ and $\boldsymbol{E} = \tilde{\boldsymbol{Q}} - \boldsymbol{Q}$.

Proof. For $\|\boldsymbol{z}(t)\|$, we have the bound

$$\|\boldsymbol{z}(t)\| \le \|\boldsymbol{E}\| \int_0^t \beta(u)\,\mathrm{d}u, \qquad t \ge 0$$

(see [4, Equation (2.5)]). This implies that

$$\sup_{t \ge 0} \|\boldsymbol{z}(t)\| \le \|\boldsymbol{E}\| \int_0^\infty \beta(u)\,\mathrm{d}u. \tag{1}$$

Note that the submultiplicativity property holds for $\beta(t)$: for all $s, t \ge 0$, $\beta(s+t) \le \beta(s)\beta(t)$; see e.g. [1, Chapter 2]. Since $\beta(s) < 1$ for all $s > 0$,

$$\beta(t) \le \exp\left(-\left\lfloor \frac{t}{\tau_1} \right\rfloor\right), \qquad t \ge 0, \tag{2}$$

the inequality being strict if $0 \le \lfloor t/\tau_1 \rfloor < t/\tau_1$. It follows that

$$\int_0^\infty \beta(u)\,\mathrm{d}u < \int_0^\infty \exp\left(-\left\lfloor \frac{u}{\tau_1} \right\rfloor\right) \mathrm{d}u = \tau_1(1 + \mathrm{e}^{-1} + \mathrm{e}^{-2} + \cdots) = \frac{\mathrm{e}\tau_1}{\mathrm{e} - 1}.$$

This, together with (1), proves the theorem.

Theorem 1 shows that the quantity $\mathrm{e}\tau_1/(\mathrm{e} - 1)$ can be viewed as a condition number with respect to perturbations of the generator. Now we shall bound τ_1 in terms of the eigenvalues of $\boldsymbol{Q}$.

Denote the eigenvalues of $\boldsymbol{Q}$ by λ_m, $m = 0, 1, \ldots, N$; choose $\lambda_0 = 0$. Notice that, since $X(t)$ has a unique stationary distribution, the zero eigenvalue has multiplicity 1. Set $\lambda = \min_{1 \le m \le N} |\operatorname{Re} \lambda_m|$; this quantity is the spectral gap of $\boldsymbol{Q}$. The spectral gap is equal to the decay parameter of $X(t)$, that is,

$$\lambda = \sup\{\gamma > 0 : \|\boldsymbol{p}(t) - \boldsymbol{\pi}\| = O(\exp(-\gamma t)) \text{ as } t \to \infty \text{ for all } \boldsymbol{p}(0)\}.$$

This, in combination with (2) and the inequalities

$$\exp\left(-\left\lfloor \frac{t}{\tau_1} \right\rfloor\right) \le \exp\left(1 - \left(\frac{t}{\tau_1}\right)\right) \quad \text{and} \quad \|\boldsymbol{p}(t) - \boldsymbol{\pi}\| \le 2\beta(t),$$

yields that

$$\frac{1}{\lambda} \le \tau_1. \tag{3}$$

Suppose now that $X(t)$ is reversible, that is, it is irreducible and the *detailed balance conditions*

$$\pi_i q_{ij} = \pi_j q_{ji}$$

are satisfied for all $i, j \in \mathcal{S}$. This assumption implies that all the eigenvalues of $\boldsymbol{Q}$ are real. Define the average hitting time, τ_0, by

$$\tau_0 = \sum_{i\in\mathcal{S}} \sum_{j\in\mathcal{S}} \pi_i \pi_j \,\mathrm{E}_i\, T_j,$$

where $\mathrm{E}_i\, T_j$ is the mean first hitting time for the state j if the chain starts from the state i. The eigentime identity (see [1, Chapter 3]) gives

$$\sum_{j\in\mathcal{S}} \pi_j \,\mathrm{E}_i\, T_j = \sum_{m\geq 1} \frac{1}{|\lambda_m|}$$

for each $i \in \mathcal{S}$; thus, we obtain that

$$\tau_0 = \sum_{m\geq 1} \frac{1}{|\lambda_m|}. \tag{4}$$

It follows from the results of [1, Chapter 4] that

$$\tau_1 \leq 66\tau_0. \tag{5}$$

This, together with (3) and (4), implies that, for a reversible chain,

$$\frac{1}{\lambda} \leq \tau_1 \leq 66 \sum_{m\geq 1} \frac{1}{|\lambda_m|} \leq \frac{66N}{\lambda}. \tag{6}$$

We have thus proved the following. If a Markov chain with a unique stationary distribution is well conditioned (i.e. if τ_1 is small), then all nonzero eigenvalues of its generator are well separated from 0 (since the spectral gap is large). If the chain is reversible and all nonzero eigenvalues are well separated from 0 (i.e. if the spectral gap is large), then the chain is well conditioned. It should be noted that the latter remark treats N as fixed, while many interesting chains are members of families which depend on N. To investigate the stability of such chains via (6), it is useful to know how λ depends on N. For increasing N, the inequalities in (6) may produce inaccurate quantitative estimates of stability, but, as we shall see in Section 3, they can give good qualitative results.

3. An example

As an example, consider the M/M/N/0 queue (a Markovian queueing system with N servers and no waiting rooms for customers). The corresponding stochastic model is a birth–death process, $X(t)$, over $\mathcal{S}$ with birth rates $b_n = \alpha,\ n = 0, \dots, N-1$, and death rates $d_n = \mu n$, $n = 1, \dots, N$. Note that a birth–death process is a reversible Markov chain. Define

$$\begin{aligned}
\kappa_1 &= \lambda^{-1}(\log(C) + 1), \quad \text{where } C = \sqrt{\max_{i\in\mathcal{S}} \frac{1}{\pi_i} - 1},\\
\kappa_2 &= \frac{66\mathrm{e}\tau_0}{\mathrm{e}-1},\\
\kappa_3 &= \frac{66\mathrm{e}N}{(\mathrm{e}-1)\lambda}.
\end{aligned}$$

It follows from Corollary 2.1, Remarks 2.1 and 2.2 and the inequality (3.8) of [4] that, for reversible $X(t)$, the following inequality holds:

$$\sup_{t\geq 0} \|z(t)\| \leq \kappa_1 \|\boldsymbol{E}\|;$$

Table 1: Numerical experiments.

N	α	μ	κ_1	κ_2	κ_3
3	1	0.2	4.46	2.898×10^2	4.946×10^2
		$\frac{1}{3}$	3.36	2.610×10^2	4.698×10^2
		0.5	2.43	2.253×10^2	4.147×10^2
		$\frac{2}{3}$	2.22	1.948×10^2	3.599×10^2
		1	2.07	1.501×10^2	2.750×10^2
		2	1.54	0.859×10^2	1.520×10^2
8	1	0.125	18.95	1.000×10^3	3.341×10^3
		0.2	13.68	9.207×10^2	3.301×10^3
		0.25	10.63	8.340×10^2	2.978×10^3
		0.5	9.02	5.056×10^2	1.663×10^3
20	1	0.025	82.41	2.644×10^3	1.009×10^4
		0.05	107.09	3.296×10^3	2.088×10^4
		0.1	59.01	3.000×10^3	2.054×10^4
		0.15	43.52	2.217×10^3	1.392×10^4
		0.2	42.87	1.726×10^3	1.044×10^4
		0.4	35.64	9.041×10^2	5.221×10^3
		0.8	25.70	4.611×10^2	2.610×10^3

if $C > 1$, then the inequality is strict. Thus, κ_1 is a condition number with respect to perturbations of the generator. The quantities κ_2 and κ_3 are also condition numbers; for κ_2 this follows from Theorem 1 and (5), while for κ_3 this follows from Theorem 1 and (6). The condition numbers κ_1, κ_2 and κ_3 were calculated for the M/M/N/0 queue; the results are given in Table 1. The parameter values are taken from [5].

Table 1 shows that the qualitative behaviour of κ_2 and κ_3 is in very good agreement with that of κ_1, which is much better quantitatively. But κ_2 and κ_3 are not supposed to give precise quantitative bounds. Still, it is interesting to note that, if the constant 66 in (5) could be reduced to 1 (this conjecture appears in [1, Chapter 4]), then $\kappa_2/66$ would be a condition number, which is sometimes smaller than κ_1. For example, for $N = 3$, $\alpha = 1$, $\mu = 2$, we have $\kappa_1 \approx 1.54$ and $\kappa_2/66 \approx 1.30$.

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