THEORY PROBAB. APPL.
Vol. 50, No. 2, pp. 319–326

STABILITY ESTIMATES FOR FINITE HOMOGENEOUS CONTINUOUS-TIME MARKOV CHAINS*

A. YU. MITROPHANOV†

(*Translated by M. V. Khatuntseva*)

Abstract. This paper obtains new stability estimates on infinite time interval and limit stability estimates for a finite homogeneous continuous-time Markov chain with a unique stationary distribution. The connection between the stability of the Markov chain under perturbation of the generator and the rate of convergence to stationarity is considered. Markov chains with a strongly accessible state are given special attention.

Key words. continuous-time Markov chain, stability estimates under perturbations, ergodicity coefficient, exponential convergence, spectral gap, strongly accessible state

DOI. 10.1137/S0040585X97981718

1. Introduction.

By *stability* of a continuous-time Markov chain we mean its quality of maintaining the approximate values of its probabilistic characteristics under small changes in the parameters (the generator and the initial distribution). In the case when Markov chains are used as models of real processes, the parameters defining the chain are given with some error. This leads to the necessity of obtaining quantitative stability estimates for the Markov chain, i.e., estimates of the magnitude of possible changes of the characteristics under the given parameters of perturbation. The stability estimates on the infinite time interval and limit stability estimates are of special interest (see [1]). Stability estimates for continuous-time Markov chains were considered in [2], [3], [4], [5], [6], [7], [8], [9], [10], and [11]. This paper proves new inequalities, permitting us to estimate the stability of homogeneous continuous-time Markov chains with a finite state space. The obtained estimates show a connection between stability and the characteristics of the Markov chains such as the rate of convergence to the stationary distribution, number of states, the magnitude of the spectral generator gap, and absolute value of the entries of the generator.

Let $X = \{X(t),\, t \geqq 0\}$ and $\widetilde{X} = \{\widetilde{X}(t),\, t \geqq 0\}$ be continuous-time Markov chains with state space $\mathcal{S} = \{0, 1, \dots, N\}$ $(1 \leqq N < \infty)$ and generators Q and $\widetilde{Q} = Q + E$, respectively. We denote by $p(t)$ and $\widetilde{p}(t)$ the vectors of state probabilities (*distributions*) of the chains X and $\widetilde{X}$ at time $t \geqq 0$ (all vectors without the transpose sign are column vectors). In what follows we assume that the chain X has a unique stationary distribution $\pi = (\pi_j)$.

As a measure of closeness of two distributions $p^{(1)}$ and $p^{(2)}$ on the set $\mathcal{S}$ we shall use $d(p^{(1)}, p^{(2)})$, which we define to be equal to a variation distance between the respective probability measures $p^{(1)}(\mathcal{A})$, $p^{(2)}(\mathcal{A})$ $(\mathcal{A} \subseteq \mathcal{S})$ [12]:

$$d(p^{(1)}, p^{(2)}) = \max \sum_i \bigl|p^{(1)}(\mathcal{A}_i) - p^{(2)}(\mathcal{A}_i)\bigr|,$$

where the maximum is taken over all possible partitions $\{\mathcal{A}_i\}$ of the set $\mathcal{S}$. By $\|\cdot\|$ we denote the l_1-norm (absolutely entry sum) for vectors and the ∞-norm (maximum absolute row sum) for matrices. The equality $d(p^{(1)}, p^{(2)}) = \|p^{(1)} - p^{(2)}\|$ holds. In what follows we shall need the relation

$$d(p^{(1)}, p^{(2)}) = 2 \max_{\mathcal{A} \subseteq \mathcal{S}} \bigl|p^{(1)}(\mathcal{A}) - p^{(2)}(\mathcal{A})\bigr|. \tag{1}$$

*Received by the editors November 13, 2001; revised October 7, 2004.
http://www.siam.org/journals/tvp/50-2/98171.html

†Faculty of Mathematics and Mechanics, Saratov State University, Astrakhanskaya St., 83, 410012 Saratov, Russia (MitrophanovAY@info.sgu.ru).

(Concerning (1) we note that some authors call variation distance the quantity which is twice as small as our $d(\cdot,\cdot)$.)

The main goal of this paper is to obtain upper bounds on $\|\widetilde{p}(t) - p(t)\|$ and $\|\widetilde{\pi} - \pi\|$, where $\widetilde{\pi}$ is a stationary distribution of the chain $\widetilde{X}$ (generally speaking, it is not necessarily unique). In inequalities permitting us to estimate the stability of the distribution of the chain X, the right-hand side is of the form $\varkappa\|E\|$. The value $\varkappa$ characterizes the sensitivity of the chain X to perturbations of the generator; following [13] we call similar values *condition numbers*. If a condition number is large, then we call the chain *ill-conditioned*, and if the condition number is small, we call the chain *well-conditioned* (with respect to the given condition number).

2. Stability and the rate of convergence to the stationary process. In what follows we shall need "the notation of the ergodicity coefficient of a matrix" (generated by the vector l_1-norm). The ergodicity coefficient $\tau_1(B)$ of a real square matrix B is defined in the following way:

$$\tau_1(B) = \sup_{\|v\|=1,\, v^{\mathrm{T}}h=0} \|B^{\mathrm{T}}v\|,$$

where v are vectors with real entries and h is the vector of all ones. For $\tau_1(B)$ the following relation holds:

$$\tau_1(B) = \frac{1}{2}\max_{i,j\in\mathcal{S}} \|B^{\mathrm{T}}(e_i - e_j)\|; \tag{2}$$

here and in what follows e_k is the vector whose kth entry equals 1, and other entries equal 0 (see [14]). We recall the property of the ergodicity coefficient $\beta_t := \tau_1(e^{tQ})$, which we shall use in what follows: $\beta_t < 1$ for all $t > 0$. This property follows from (2) and from the fact that the chain X has a unique stationary distribution. In addition, obviously, $\lim_{t\to\infty}\beta_t = 0$.

Taking into account the definition of the ergodicity coefficient we have

$$\begin{aligned}\|\widetilde{p}(t) - p(t)\| &= \|\widetilde{P}^{\mathrm{T}}(t)\widetilde{p}(0) - P^{\mathrm{T}}(t)\,p(0)\| \leqq \|P^{\mathrm{T}}(t)(\widetilde{p}(0) - p(0))\| + \|D^{\mathrm{T}}(t)\,\widetilde{p}(0)\| \\ &\leqq \|\widetilde{p}(0) - p(0)\|\,\beta_t + \|D(t)\|,\end{aligned} \tag{3}$$

where $\widetilde{P}(t) = e^{t\widetilde{Q}}$, $P(t) = e^{tQ}$, $D(t) = \widetilde{P}(t) - P(t)$. Hence,

$$\sup_{t\geqq 0}\|\widetilde{p}(t) - p(t)\| \leqq \|\widetilde{p}(0) - p(0)\| + \sup_{t\geqq 0}\|D(t)\|.$$

In addition, if $\widetilde{\pi}$ is a stationary distribution of the chain $\widetilde{X}$, then, assuming $\widetilde{p}(0) = \widetilde{\pi}$ in (3) and passing to the limit as $t\to\infty$, we obtain $\|\widetilde{\pi} - \pi\| \leqq \sup_{t\geqq 0}\|D(t)\|$. In this situation we are interested in uniform in $t \geqq 0$ estimates of the value $\|D(t)\|$.

In [4] the inequality

$$\|\widetilde{p}(t) - p(t)\| \leqq \|\widetilde{p}(0) - p(0)\|\,\beta_t + \|E\|\int_0^t \beta_u\,du, \qquad t \geqq 0, \tag{4}$$

is proved; this inequality implies the estimate

$$\|D(t)\| < t\,\|E\|, \qquad t > 0. \tag{5}$$

Relation (5) permits us to estimate the stability of the distribution of the chain X only at finite time intervals. Stability estimates for infinite intervals can be obtained if we use the exponential estimates of the rate of convergence $p(t)$ to π as $t\to\infty$.

Let $b, C > 0$ be such that for all $p(0)$ the inequality

$$\|p(t) - \pi\| \leqq Ce^{-bt}, \qquad t \geqq 0, \tag{6}$$

holds. Note that (6) implies $C \geqq 2(1 - \min_{j\in\mathcal{S}} \pi_j) \geqq 1$; in the right-hand side the equality is achieved only for chains with two states and a unique stationary distribution. From (4) and (6) the estimate

$$\sup_{t\geqq 0} \|D(t)\| \leqq b^{-1}(\log C + 1)\|E\| \tag{7}$$

follows; if $C > 1$, then the inequality in (7) is strict; see [4]. Inequality (7) differs from the results obtained earlier by the logarithmic dependence of the right-hand side on C; results of other authors give estimates with linear dependence (see [4]). In this section we give a new stability estimate with logarithmic dependence of the right-hand side on C. The proof is based on the results of Anisimov [11].

LEMMA 1. *The inequality*

$$\sup_{t\geqq 0} \|D(t)\| \leqq \inf_{t>0} \gamma_t \|E\|, \tag{8}$$

where $\gamma_t = t/(1-\beta_t)$, *holds.*

Proof. Let

$$\varphi_t = \max_{i,j\in\mathcal{S},\mathcal{A}\subseteq\mathcal{S}} |p(i,t,\mathcal{A}) - p(j,t,\mathcal{A})|, \qquad \psi_t = \max_{i\in\mathcal{S},\mathcal{A}\subseteq\mathcal{S}} |p(i,t,\mathcal{A}) - \widetilde{p}(i,t,\mathcal{A})|,$$

where $p(i,t,\mathcal{A})$, $\widetilde{p}(i,t,\mathcal{A})$ are the transition functions for chains X and $\widetilde{X}$, respectively ($i \in \mathcal{S}$, $t \geqq 0$, $\mathcal{A} \subseteq \mathcal{S}$). If $\varphi_s < 1$ for some $s > 0$, then, applying Theorem 2 of [11], we obtain

$$\psi_t \leqq \frac{\sup_{u\leqq s} \psi_u}{1-\varphi_s}\bigl(1 - \varphi_s^{[t/s]+1}\bigr), \qquad t \geqq 0, \tag{9}$$

where $[\,\cdot\,]$ denotes the integer part. Using (1), it is not difficult to see that $\varphi_t = \beta_t$ and $\psi_t = \|D(t)\|/2$. Hence, $\sup_{u\leqq s} \psi_u < s\|E\|/2$. These relations, inequality (9), and the fact that $\beta_t < 1$ for all $t > 0$ imply the statement of the lemma.

THEOREM 1. *Let* $b > 0$ *and* $C \geqq 1$ *be such that* (6) *holds. Then*

$$\sup_{t\geqq 0} \|D(t)\| \leqq \inf_{0<y<1} \frac{b^{-1}\log(C/y)}{1-y}\,\|E\|. \tag{10}$$

Proof. Taking into account (2) and applying the triangle inequality we obtain

$$\beta_t \leqq \max_{j\in\mathcal{S}} \|P^{\mathrm{T}}(t)\,e_j - \pi\| \leqq Ce^{-bt}, \qquad t \geqq 0. \tag{11}$$

Let t_0 be such that $Ce^{-bt_0} = 1$. Since $C \geqq 1$, it follows that $t_0 \geqq 0$. By (8) and (11) we have

$$\sup_{t\geqq 0} \|D(t)\| \leqq \frac{t\|E\|}{1-Ce^{-bt}}, \qquad t > t_0. \tag{12}$$

Let $y = Ce^{-bt}$; $y \in (0,1)$ for $t > t_0$. Then (12) can be written as

$$\sup_{t\geqq 0} \|D(t)\| \leqq \frac{b^{-1}\log(C/y)}{1-y}\,\|E\|, \qquad y \in (0,1).$$

The theorem is proved.

Remark 1. Different approaches to obtaining estimates of the form (6) are discussed in [4].

The behavior of the function $f(y) := \log(C/y)/(1-y)$ on the interval $(0,1)$ is of interest. The following proposition is valid.

PROPOSITION 1. *If* $C > 1$, *then* $f(y)$ *has one critical point* y_0 *on* $(0,1)$. *At this point* $f(y)$ *has a minimum, and* $\lim_{y\downarrow 0} f(y) = \lim_{y\uparrow 1} f(y) = \infty$. *We can find the point* y_0 *from the equation* $C = y\exp((1-y)/y)$. *If* $C = 1$, *then* $f(y)$ *monotonically decreases on* $(0,1)$, $\lim_{y\downarrow 0} f(y) = \infty$, *and* $\lim_{y\uparrow 1} f(y) = 1$.

3. Stability estimates and the entries of the generator. Here we give stability estimates for which the condition number can be explicitly expressed in terms of the entries q_{ij} of the generator Q. As in the previous section, Lemma 1 is the base for the obtained results. An especially interesting and easily applied result was proved for Markov chains with a strongly accessible state.

DEFINITION 1. *If there exists $k_0 \in \mathcal{S}$ such that $q_{ik_0} > 0$ for all $i \in \mathcal{S} \setminus \{k_0\}$, then we say that the chain X has a* strongly accessible state k_0 (*similarly to the discrete-time case*; *see* [15]).

Lemma 1 shows that $\sup_{t\geqq 0} \|D(t)\|$ can be estimated from above by $\lim_{t\to\infty} \gamma_t \|E\|$ and $\lim_{t\to 0} \gamma_t \|E\|$ if the limits exist and are finite. It is not difficult to see that $\lim_{t\to\infty} \gamma_t = \infty$. Theorem 2 explains the behavior of γ_t as $t \to 0$.

DEFINITION 2. *We say that* assumption A is satisfied *for a pair of states* $(i_0, j_0) \in \mathcal{S} \times \mathcal{S}$ *($i_0 \neq j_0$) if the following statements are valid simultaneously*:

(a) $q_{i_0 j_0} = q_{j_0 i_0} = 0$;

(b) *there is no $k \in \mathcal{S} \setminus \{i_0, j_0\}$ such that $q_{i_0 k} > 0$ and $q_{j_0 k} > 0$.*

THEOREM 2. *If for some pair of states assumption* A *is satisfied, then* $\lim_{t\to 0} \gamma_t = \infty$; *otherwise* $0 < \lim_{t\to 0} \gamma_t < \infty$,

$$\lim_{t\to 0} \gamma_t = \xi^{-1}, \qquad \xi := \frac{1}{2} \min_{i,j\in\mathcal{S}, i\neq j} f_{ij},$$

$$f_{ij} := |q_{ii} + q_{jj} - q_{ij} - q_{ji}| - \sum_{k\in\mathcal{S}\setminus\{i,j\}} |q_{ik} - q_{jk}|, \qquad i \neq j.$$

Proof. Denote by $p_{ij}(t)$ the entries of the matrix $P(t)$. For $t \to 0$ we have $p_{ij}(t) = q_{ij} t + o(t)$, $i \neq j$, $p_{ii}(t) = 1 + q_{ii} t + o(t)$. From here, for $t \to 0$,

$$\sum_{k\in\mathcal{S}} \bigl|p_{ik}(t) - p_{jk}(t)\bigr| = 2 - |q_{ii} + q_{jj} - q_{ij} - q_{ji}|\, t + \sum_{k\in\mathcal{S}\setminus\{i,j\}} |q_{ik} - q_{jk}|\, t + o(t), \qquad i \neq j.$$

Taking into account (2), as $t \to 0$ we obtain

$$1 - \beta_t = \frac{t}{2} \min_{i,j\in\mathcal{S}, i\neq j} \bigl\{f_{ij} + \varepsilon_{ij}(t)\bigr\}, \qquad \lim_{t\to 0} \varepsilon_{ij}(t) = 0.$$

For some pairs of indices $(i_k, j_l) \in \mathcal{S} \times \mathcal{S}$ ($i_k \neq j_l$) the equality $\frac{1}{2} f_{i_k j_l} = \xi$ holds; hence, there exists $t_0 > 0$ such that

$$\frac{1 - \beta_t}{t} = \xi + \frac{1}{2} \min_{(i_k, j_l)} \varepsilon_{i_k j_l}(t), \qquad t < t_0. \tag{13}$$

If for some pair (i_0, j_0) assumption A is satisfied, then $\xi = f_{i_0 j_0} = 0$, since $\sum_{j\in\mathcal{S}} q_{ij} = 0$ for all $i \in \mathcal{S}$. If assumption A is not satisfied for any pair of states, then $f_{ij} > 0$ for all $i, j \in \mathcal{S}$ ($i \neq j$). Passing in (13) to the limit as $t \to 0$, we arrive at the statement of the theorem.

Remark 2. Assumption A is not satisfied for any pair of states if, for example, the chain X has a strongly accessible state. Note that the existence of a strongly accessible state is not necessary for satisfying the relation $\lim_{t\to 0} \gamma_t < \infty$. For example, we consider the case when $\mathcal{S} = \{0, 1, 2, 3\}$ and the one-jump transition structure of the chain X for one jump is the following: $0 \leftrightarrow 1$, $1 \leftrightarrow 2$, $2 \leftrightarrow 3$, $3 \leftrightarrow 0$ (here the notation $i \leftrightarrow j$ denotes that $q_{ij} > 0$ and $q_{ji} > 0$); other transitions are impossible. By Theorem 2, for such a chain we have $\lim_{t\to 0} \gamma_t < \infty$, although there are no strongly accessible states.

THEOREM 3. *If the chain X has a strongly accessible state*, *then*

$$\delta := \sum_{j\in\mathcal{S}} \min_{i\in\mathcal{S}\setminus\{j\}} q_{ij} > 0$$

and

$$\sup_{t\geqq 0} \bigl\|D(t)\bigr\| \leqq \delta^{-1} \|E\|. \tag{14}$$

Proof. For β_t the following representation takes place (see [16]):

$$\beta_t = 1 - \min_{i,j\in\mathcal{S}} \sum_{k\in\mathcal{S}} \min\bigl\{p_{ik}(t), p_{jk}(t)\bigr\}.$$

Hence,

$$\beta_t \leqq 1 - \sum_{j\in\mathcal{S}} \min_{i\in\mathcal{S}} p_{ij}(t). \tag{15}$$

In view of (8) and (15), we obtain

$$\sup_{t\geqq 0} \bigl\|D(t)\bigr\| \leqq \inf_{t>0} \frac{t\|E\|}{\sum_{j\in\mathcal{S}} \min_{i\in\mathcal{S}} p_{ij}(t)}. \tag{16}$$

Since $\lim_{t\to 0} p_{ii}(t) = 1$ and $\lim_{t\to 0} p_{ij}(t) = 0$ for $i \neq j$, there exists $t_0 > 0$ such that

$$\sum_{j\in\mathcal{S}} \min_{i\in\mathcal{S}} p_{ij}(t) = \sum_{j\in\mathcal{S}} \min_{i\in\mathcal{S}\setminus\{j\}} p_{ij}(t), \qquad t \leqq t_0.$$

If $i \neq j$, then for $t \to 0$ we have $p_{ij}(t) = q_{ij}t + o(t)$. Hence,

$$\lim_{t\to 0} \sum_{j\in\mathcal{S}} \min_{i\in\mathcal{S}} \frac{p_{ij}(t)}{t} = \delta.$$

From this expression and inequality (16) the statement of the theorem follows.

Remark 3. The proof of Theorem 3 implies that if X has a strongly accessible state, then $\xi^{-1} \leqq \delta^{-1}$.

We note that Markov chains with a strongly accessible state are interesting with their applications. As an example we consider a chemical reaction between substances $T_0, T_1, \dots, T_N$, in which a molecule of species T_0 can be in one of $N+1$ states: free, or bound with a molecule of species T_l, $l = 1, \dots, N$. The molecule T_0 passes from a free to a bound state if it bounds with one of the particles T_l. Suppose that the molecule T_0 can pass to a free state again. Transitions from one bound state to another reflect transformations of bound molecules (which occur, for example, as a result of the action of an enzyme). Similar reactions are often met in molecular biology [17]. To describe changes of the state of the molecule T_0 one can use the continuous-time Markov chain with the state space $\mathcal{S}$ having the strongly accessible state 0. The state 0 in this model corresponds to the free molecule T_0 and the other states correspond to T_0 bounded with one of the molecules T_l, $l = 1, \dots, N$ (a justification for the application of continuous-time Markov chains for modeling chemical reactions can be found, for example, in [18]).

It is interesting to compare the statement of Theorem 3 with the stability estimate for the stationary distribution of a discrete-time Markov chain obtained in [19]. Let Y and $\widetilde{Y}$ be finite homogeneous discrete-time Markov chains with the state space $\mathcal{S}$ having unique stationary distributions ρ and $\widetilde{\rho}$, respectively. Denote by P and $\widetilde{P}$ the transition probability matrices for chains Y and $\widetilde{Y}$. Let $\nu_{k_0} := \min_{i\in\mathcal{S}\setminus\{k_0\}} p_{ik_0} > 0$ for some k_0; here p_{ij} are entries of the matrix P. In this case the following inequality is valid (see [19]):

$$\|\widetilde{\rho} - \rho\| \leqq \nu_{k_0}^{-1} \|\widetilde{P} - P\|.$$

The condition $\nu_{k_0} > 0$ for some k_0 is the condition for the existence of a strongly accessible state k_0 for the discrete-time chain Y.

Passing to the continuous-time case we suppose that the chain X has a strongly accessible state. Then from (14) it follows that for an arbitrary stationary distribution $\widetilde{\pi}$ of the chain $\widetilde{X}$ and a unique stationary distribution π of the chain X the estimate

$$\|\widetilde{\pi} - \pi\| \leqq \delta^{-1}\|E\|$$

is valid. Note that $\delta \geqq \widehat{\delta} := \max_{j\in\mathcal{S}} \min_{i\in\mathcal{S}\setminus\{j\}} q_{ij} > 0$. Thus the stability of the stationary distribution of the Markov chain with strongly accessible state is characterized with the help of the values of the nondiagonal entries of the transition matrix (in the discrete-time case) or with the help of the values of the intensity transition (in the continuous-time case). In this case the dependence of the condition number $\min_{k_0} \nu_{k_0}^{-1}$ on entries of the matrix P is analogous to the dependence of $\widehat{\delta}^{-1}$ on the entries of Q.

4. Stability estimates and eigenvalues of the generator. Denote by λ_m, $m = 1, \dots, N+1$, the eigenvalues of Q (each of the distinct eigenvalues is counted according to its multiplicity). Set $\lambda_1 = 0$. Since the chain X, by assumption, has a unique stationary distribution, its zero eigenvalue has multiplicity 1; real parts of other eigenvalues are negative. Set $\lambda = \min_{2\leqq m\leqq N+1} |\operatorname{Re}\lambda_m|$; this value is called the *spectral gap* of the generator Q. For the chain X the spectral gap is equal to the *convergence parameter*:

$$\lambda = \sup\left\{\alpha > 0\colon\ \left\|p(t) - \pi\right\| = O(e^{-\alpha t}) \text{ as } t \to \infty \text{ for all } p(0)\right\}$$

(see [20], where a method of estimating λ is considered). Thus the value λ is a characteristic of the rate of the exponential convergence to the stationary distribution for the chain X. In this situation the stability estimates for the chain X under the generator perturbations expressed in terms of λ are of interest.

Set $\mu_m(t) = e^{\lambda_m t}$, where $1 \leqq m \leqq N+1$, $t \geqq 0$. The quantities $\mu_m(t)$ are the eigenvalues of the matrix $P(t)$. Denote $\mu(t) = \max_{2\leqq m\leqq N+1} |\mu_m(t)|$.

PROPOSITION 2. *The following estimate holds*:

$$\lambda^{-1} \leqq \inf_{t>0} \gamma_t.$$

Proof. Using the known property of the ergodicity coefficient (see, for example, [21]), we obtain $\mu(t) \leqq \beta_t$, $t > 0$. Since $\mu(t) = e^{-\lambda t}$, it follows that

$$\lambda^{-1} = \inf_{t>0} \frac{t}{1 - e^{-\lambda t}} \leqq \inf_{t>0} \gamma_t.$$

The proposition is proved.

COROLLARY 1. *If the chain X has a strongly accessible state*, *then $\delta \leqq \xi \leqq \lambda$.*

Proposition 2 shows that if the chain X is well-conditioned (with respect to the condition number $\inf_{t>0} \gamma_t$), then it converges fast to the stationary process. Note that the stability estimates in Theorems 1 and 3 are obtained by roughening inequality (8). Hence, if the chain is well-conditioned with respect to the condition number in (10) and also with respect to ξ^{-1} and δ^{-1}, then the chain is well-conditioned with respect to $\inf_{t>0} \gamma_t$. Theorem 3 and Corollary 1 permit us to reduce the analysis of stability and rate of convergence for Markov chains with a strongly accessible state to finding the quantity δ.

In the proof of the following theorem we use the notation of a *group inverse matrix $B^{\#}$* of a real square matrix B. The matrix $B^{\#}$ satisfies the following relations: $BB^{\#}B = B$, $B^{\#}BB^{\#} = B^{\#}$, and $BB^{\#} = B^{\#}B$; if $B^{\#}$ exists, then it is unique. The notion of the group inverse is widely applied in the theory of finite discrete-time Markov chains; see [13], [22].

THEOREM 4. *For an arbitrary stationary distribution $\widetilde{\pi}$ of the chain $\widetilde{X}$ and a unique stationary distribution π of the chain X the following estimate is valid*:

$$\|\widetilde{\pi} - \pi\| \leqq N\rho^{-1}\|E\|,$$

where $\rho = \min_{2\leqq m\leqq N+1} |\lambda_m|$.

Proof. The distributions π and $\widetilde{\pi}$ are the stationary distributions of discrete-time Markov chains with state space, the state space $\mathcal{S}$, and the transition matrices $P(t)$ and $\widetilde{P}(t)$, $t > 0$, respectively. Moreover, for any $t > 0$, π is a unique stationary distribution corresponding to $P(t)$. These facts, together with relation (4) of [23] and our inequality (5), give the estimate

$$\|\widetilde{\pi} - \pi\| \leqq \inf_{t>0} \tau_1\bigl(A^{\#}(t)\bigr)\, t\|E\|, \tag{17}$$

where $A(t) = I - P(t)$, I is the identity matrix.

Inequality (10) of [23] implies

$$\tau_1(A^{\#}(t)) \leqq \sum_{m=2}^{N+1} \frac{1}{1-\mu_m(t)}.$$

Since $\tau_1(\cdot)$ is nonnegative, it follows that

$$\tau_1(A^{\#}(t)) \leqq \frac{N}{\min_{2\leqq m\leqq N+1} |1-\mu_m(t)|}. \tag{18}$$

From (17) and (18)

$$\|\widetilde{\pi} - \pi\| \leqq \inf_{t>0} \frac{Nt\|E\|}{\min_{2\leqq m\leqq N+1} |1-\mu_m(t)|} \leqq \lim_{t\to 0} \frac{N\|E\|}{\min_{2\leqq m\leqq N+1} |1-\mu_m(t)|/t} \tag{19}$$

follows. Passing to the limit as $t \to 0$ in the right-hand side of (19) and taking into account that

$$\lim_{t\to 0} t^{-1}\bigl(1-\mu_m(t)\bigr) = \lambda_m,$$

we arrive at the statement of the theorem.

COROLLARY 2. $\|\widetilde{\pi} - \pi\| \leqq N\lambda^{-1}\|E\|$.

Acknowledgment. The author is deeply grateful to the referee for valuable remarks which have allowed the author to improve the paper significantly.

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