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REVERSIBLE MARKOV CHAINS AND SPANNING TREES

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Abstract

For a finite continuous-time Markov chain, we prove a sufficient condition for reversibility in terms of a spanning tree and the corresponding fundamental cycles of the chain's transition graph. We demonstrate how this sufficient condition can be used to construct reversible Markov chains given the transition graph.

Keywords: Markov chain; reversibility; spanning tree; fundamental cycle; stationary distribution

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1. Introduction

In this paper we consider reversible continuous-time Markov chains; for a general background on Markov chains in continuous time, see, for example, [5]. We begin with a simple example. Let $X(t),\ t \geq 0$, be a birth–death process taking values in a finite set $\mathcal{S} = \{0, 1, \dots, N\}$, with birth rates $\{\lambda_i\}$ and death rates $\{\mu_i\}$, $i \in \mathcal{S}$. The λ_i and μ_i are all positive except $\lambda_N = \mu_0 = 0$. Let $\pi(i)$ denote the stationary probabilities of $X(t)$. We then have the following well-known formulae (see [5]):

$$\pi(i) = \pi(0) \prod_{k=0}^{i-1} \frac{\lambda_k}{\mu_{k+1}}, \qquad i = 1, \dots, N, \tag{1}$$

$$\pi(0) = \left(1 + \sum_{i=1}^{N} \prod_{k=0}^{i-1} \frac{\lambda_k}{\mu_{k+1}}\right)^{-1}. \tag{2}$$

Denoting by $q(i, j)$ the entries of the generator of $X(t)$, we have

$$\begin{aligned} q(i, i+1) &= \lambda_i, \\ q(i, i-1) &= \mu_i, \\ q(i, j) &= 0, \qquad j \neq i, i-1, i+1. \end{aligned}$$

It follows from (1) and (2) that $\pi(i)q(i, j) = \pi(j)q(j, i)$ for all $i, j \in \mathcal{S}$. For an irreducible Markov chain, this relation is the definition of reversibility; thus, birth–death processes are reversible continuous-time Markov chains. Obviously, for the process defined above, changing the λ_i or μ_i will not affect reversibility. However, if the transition structure of a Markov chain is more complex than that of a birth–death process, whether the chain is reversible or not depends

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on the actual values of the transition rates. In this paper, we study conditions of reversibility for continuous-time Markov chains.

The notion of reversibility has proved to be very important for both the theory and the applications of Markov chains; see [1], [6]. One of the reasons for this is the fact that many problems which are difficult or even intractable in the general case can be (sometimes relatively easily) solved for reversible Markov chains (see [1] and references cited therein). Another reason is that some modeling problems require the use of reversible Markov chains. This is the case in chemical physics since the property of reversibility is a reflection of the physical principle of detailed balance (see [9]). As an example, we point out that reversible continuous-time Markov chains are widely used in ion channel modeling as in [2], [3].

For a continuous-time Markov chain, it is indeed necessary to be able to establish that the chain is reversible directly from its transition rates. The usual way of doing this is through the use of Kolmogorov's criterion: a Markov chain is reversible if and only if, for each cycle in the state space, the product of the transition rates is independent of the direction in the cycle (see [6]). In this paper we consider a sufficient condition for reversibility based on the notions of a spanning tree and fundamental cycles of a graph. We call this the *fundamental cycle condition*. We show that a Markov chain is reversible if the product of the transition rates is independent of the direction in the cycle for all the fundamental cycles that correspond to some spanning tree of the transition graph. The proof of the fundamental cycle condition that we give here is independent of Kolmogorov's criterion, allowing us to obtain the latter as a simple corollary.

If we wish to use reversible continuous-time Markov chains for modeling the behaviour of real-world systems, we need methods of constructing reversible Markov chains. Suppose that we are given a transition graph that contains cycles. The objective is to give values to the transition rates in such a way that the resulting chain is reversible. We show that the fundamental cycle condition can be effectively used to solve this problem. It should be noted that the discrete-time counterparts of our results can be easily obtained in a similar manner.

2. The fundamental cycle condition

In this section we prove the fundamental cycle condition. We first introduce some notation and definitions. Let $\mathcal{S}$ denote an arbitrary finite set of numbers, with cardinality $n \geq 2$. Consider a homogeneous, continuous-time Markov chain, $X(t),\ t \geq 0$, with state space $\mathcal{S}$ and irreducible generator $\boldsymbol{Q}$ whose entries we denote by $q(i, j),\ i, j \in \mathcal{S}$. Since the chain $X(t)$ is irreducible, it has a unique stationary distribution, and the stationary probabilities are positive for every state. Suppose that, whenever $i \neq j$, $q(i, j) > 0$ implies that $q(j, i) > 0$. Define the transition graph, Γ, for the chain $X(t)$ as a nondirected graph with vertex set $\mathcal{S}$ and edge set $\mathcal{E}$ such that, when $i \neq j$, $(i, j) \in \mathcal{E}$ if and only if $q(i, j) > 0$. Note that the graph Γ is connected.

Let Δ be a spanning tree of Γ with edge set $\mathcal{F}$, and let $Z(t),\ t \geq 0$, be a Markov chain with transition graph Δ, the corresponding transition rates being equal to those of $X(t)$. Every spanning tree of Γ defines a system of fundamental cycles of Γ (for a background on spanning trees and fundamental cycles, see [4, Chapters 7 and 9]). For every edge $(k, l) \in \mathcal{E} \setminus \mathcal{F}$, the corresponding fundamental cycle is formed by (k, l) and the unique path in Δ from k to l.

We call the Markov chain $X(t)$ *reversible* if there exist positive numbers $\gamma(i),\ i \in \mathcal{S}$, such that, for all $i, j \in \mathcal{S}$, $\gamma(i)q(i, j) = \gamma(j)q(j, i)$; these are the well-known *detailed balance conditions*. This definition of reversibility was used by Aldous and Fill [1] (see also references cited therein); Kelly [6] gave a definition which involves time reversal and implied that the process is in equilibrium. Theorem 1.3 of [6] showed that these two definitions are closely related. To formulate the fundamental cycle condition, we need the following condition.

Condition 1. *For a cycle $i_1, i_2, \dots, i_p, i_1$ in Γ, the corresponding transition rates satisfy*

$$q(i_1, i_2)q(i_2, i_3)\cdots q(i_{p-1}, i_p)q(i_p, i_1) = q(i_1, i_p)q(i_p, i_{p-1})\cdots q(i_3, i_2)q(i_2, i_1). \quad (3)$$

We shall assume that the numbers $\gamma(i)$ are such that $\sum_{i\in\mathcal{S}} \gamma(i) = 1$. It is easy to prove that the probability vector $\boldsymbol{\gamma} = [\gamma(i)]$ is the stationary distribution of $X(t)$, that is, $\boldsymbol{Q}^\top\boldsymbol{\gamma} = \mathbf{0}$. In what follows we make use of the following important fact.

Proposition 1. (See [6, Lemma 1.5].) *If Γ is a tree, then $X(t)$ is reversible.*

Proof. If Γ is a tree, then $\Delta = \Gamma$, so we need to prove that $Z(t)$ is reversible. Let r and v be distinct states in $\mathcal{S}$, and let the path $r, j_1, j_2, \dots, j_d, v$ be the unique path from r to v in Δ. Define $\pi(i),\ i \in \mathcal{S}$, by

$$\pi(r) = \frac{1}{1 + \sum_{v\in\mathcal{S}\setminus\{r\}} p(r, v)}, \quad (4)$$

$$\pi(v) = \pi(r)p(r, v), \qquad v \in \mathcal{S} \setminus \{r\}, \quad (5)$$

where

$$p(r, v) = \frac{q(r, j_1)q(j_1, j_2)\cdots q(j_{d-1}, j_d)q(j_d, v)}{q(j_1, r)q(j_2, j_1)\cdots q(j_d, j_{d-1})q(v, j_d)}. \quad (6)$$

The probability distribution $\boldsymbol{\pi} = [\pi(i)]$ satisfies the detailed balance conditions for $Z(t)$ (for all pairs of adjacent vertices, consider the corresponding unique paths from r and use (4)–(6)). Therefore, $Z(t)$ is reversible.

Proposition 2. *If $X(t)$ is reversible, then (4)–(6) also provide expressions for the stationary probabilities of $X(t)$.*

Proof. Suppose that $X(t)$ is reversible, and consider the Markov chain $Y(t)$ whose transition graph, Ξ, is derived from Γ by deleting one of the edges in $\mathcal{E} \setminus \mathcal{F}$. Let the transition rates for the edges in Ξ be the same as for the corresponding edges in Γ. It is clear that the stationary distributions of $X(t)$ and $Y(t)$ coincide, since the stationary probabilities for $X(t)$ also satisfy the detailed balance conditions for $Y(t)$. If we continue removing the edges belonging to $\mathcal{E} \setminus \mathcal{F}$, we shall eventually obtain a Markov chain with transition graph Δ and stationary probabilities given by (4)–(6); therefore, this is also the stationary distribution of $X(t)$.

It is worth noting that, for the stationary distribution of a reversible Markov chain, more general formulae can be obtained (see [6, proof of Theorem 1.7]), but the tree formulae (4)–(6) are of particular use if we want to construct a reversible Markov chain by giving values to the transition rates for the edges in some spanning tree (see Sections 3 and 4).

Theorem 1. (The fundamental cycle condition.) *The Markov chain $X(t)$ is reversible if Condition 1 is satisfied for every cycle in some system of fundamental cycles.*

Proof. Suppose that Condition 1 is satisfied for every cycle in the system of fundamental cycles defined by Δ. As we have shown above, the chain $Z(t)$ is reversible, that is, for all pairs $(i, j) \in \mathcal{F}$, the detailed balance conditions are satisfied:

$$\pi(i)q(i, j) = \pi(j)q(j, i). \tag{7}$$

For all $k, l \in \mathcal{S}$, there is a unique path in Δ from k to l, say $k, i_1, i_2, \ldots, i_r, l$. For this path, from (7) we obtain that

$$\begin{aligned} \pi(k) &= \pi(i_1)\frac{q(i_1, k)}{q(k, i_1)} \\ &= \pi(i_2)\frac{q(i_2, i_1)q(i_1, k)}{q(i_1, i_2)q(k, i_1)} \\ &= \cdots \\ &= \pi(l)\frac{q(l, i_r)q(i_r, i_{r-1})\cdots q(i_2, i_1)q(i_1, k)}{q(i_r, l)q(i_{r-1}, i_r)\cdots q(i_1, i_2)q(k, i_1)}. \end{aligned} \tag{8}$$

Suppose now that $(k, l) \in \mathcal{E} \setminus \mathcal{F}$. By the theorem's assumption on the fundamental cycles,

$$q(k, i_1)q(i_1, i_2)\cdots q(i_{r-1}, i_r)q(i_r, l)q(l, k) = q(l, i_r)q(i_r, i_{r-1})\cdots q(i_2, i_1)q(i_1, k)q(k, l).$$

This, together with (8), gives $\pi(k)q(k, l) = \pi(l)q(l, k)$, which shows that the distribution $\boldsymbol{\pi}$ satisfies the detailed balance conditions for $X(t)$. Therefore, $X(t)$ is reversible.

Remark 1. The detailed balance conditions imply that, if $X(t)$ is reversible, then Condition 1 is satisfied for all cycles. Indeed, in this case, for the cycle $i_1, i_2, \ldots, i_p, i_1$, we have that

$$\begin{aligned} \gamma(i_1) &= \gamma(i_2)\frac{q(i_2, i_1)}{q(i_1, i_2)} \\ &= \gamma(i_3)\frac{q(i_3, i_2)q(i_2, i_1)}{q(i_2, i_3)q(i_1, i_2)} \\ &= \cdots \\ &= \gamma(i_1)\frac{q(i_1, i_p)q(i_p, i_{p-1})\cdots q(i_3, i_2)q(i_2, i_1)}{q(i_p, i_1)q(i_{p-1}, i_p)\cdots q(i_2, i_3)q(i_1, i_2)}, \end{aligned}$$

which is equivalent to (3). Therefore, Kolmogorov's criterion for reversibility (for a finite state space) is a corollary to Theorem 1.

3. Constructing reversible Markov chains

One advantage of the fundamental cycle condition is that in many cases it is much easier to check that Condition 1 holds for fundamental cycles than to check that it holds for all cycles in the graph. Consider now the problem of constructing a reversible Markov chain, $X(t)$, given a transition graph, Γ, that contains cycles. If we know a spanning tree, Δ, then we can express the ratios of transition rates for the $n-1$ edges in $\mathcal{E} \setminus \mathcal{F}$ in terms of the corresponding ratios for the edges in $\mathcal{F}$ using Condition 1 for the fundamental cycles. By Theorem 1, this will make $X(t)$ a reversible Markov chain, and the ratios of the transition rates that correspond to the edges in $\mathcal{F}$ can be arbitrary positive numbers. But is it possible to express some of the ratios for the edges that are not necessarily in $\mathcal{E} \setminus \mathcal{F}$ in terms of other such ratios if we know the spanning tree Δ? Below we treat this question in a general setting.

Let m be the cardinality of $\mathcal{E}$. The spanning tree Δ defines $\nu = m - n + 1 \geq 1$ fundamental cycles $i_{l1}, i_{l2}, \ldots, i_{l\lambda_l}, i_{l1}$ of length λ_l for $l = 1, \ldots, \nu$. Each of these cycles, together with Condition 1, gives an equation for the corresponding transition rates: the fundamental cycle $i_{l1}, i_{l2}, \ldots, i_{l\lambda_l}, i_{l1}$ has the corresponding equation

$$\frac{q(i_{l1}, i_{l2})q(i_{l2}, i_{l3}) \cdots q(i_{l\lambda_l}, i_{l1})}{q(i_{l2}, i_{l1})q(i_{l3}, i_{l2}) \cdots q(i_{l1}, i_{l\lambda_l})} = 1.$$

Denoting $K(i, j) = \log(q(i, j)/q(j, i))$, we arrive at the following system of linear algebraic equations:

$$\begin{aligned} K(i_{11}, i_{12}) + K(i_{12}, i_{13}) + \cdots + K(i_{1\lambda_1}, i_{11}) &= 0, \\ K(i_{21}, i_{22}) + K(i_{22}, i_{23}) + \cdots + K(i_{2\lambda_2}, i_{21}) &= 0, \\ &\vdots \\ K(i_{\nu 1}, i_{\nu 2}) + K(i_{\nu 2}, i_{\nu 3}) + \cdots + K(i_{\nu\lambda_\nu}, i_{\nu 1}) &= 0. \end{aligned} \tag{9}$$

Now we number all the edges in $\mathcal{E}$ consecutively from 1 to m, and let $n(i, j)$ be the number of the edge (i, j). Define the vector $\boldsymbol{k} = [k(i)]$ of dimension m as follows: for all $(i, j) \in \mathcal{E}$, set $k(n(i, j))$ equal to either $K(i, j)$ or $K(j, i)$ (the choice is free). Let $\boldsymbol{\Phi} = [\phi(v, w)]$ be a matrix of dimension $\nu \times m$ defined in the following way. For each (v, w), the entry $\phi(v, w)$ corresponds to the vth fundamental cycle (the vth equation in (9)) and the wth edge of the transition graph. Let (i, j) be the wth edge, i.e. $n(i, j) = w$. If (i, j) does not belong to the vth fundamental cycle, then neither $K(i, j)$ nor $K(j, i)$ are terms in the vth equation in (9). In this case, $\phi(v, w) = 0$. Suppose now that $K(i, j)$ is a term in the vth equation in (9). In this case, if $k(w) = K(i, j)$, then $\phi(v, w) = 1$ and, if $k(w) = K(j, i)$, then $\phi(v, w) = -1$. The system (9) can be written in the matrix–vector form

$$\boldsymbol{\Phi}\boldsymbol{k} = \boldsymbol{0}. \tag{10}$$

The rank of $\boldsymbol{\Phi}$ is equal to ν since the columns of $\boldsymbol{\Phi}$ that correspond to the edges in $\mathcal{E} \setminus \mathcal{F}$ are linearly independent (every edge in $\mathcal{E} \setminus \mathcal{F}$ belongs to only one fundamental cycle). If we find another system of ν linearly independent columns of $\boldsymbol{\Phi}$, then we can uniquely express the corresponding values $k(i)$ in terms of other such quantities, and our problem will be solved. For example, consider the first ν columns of $\boldsymbol{\Phi}$, denoted by $\boldsymbol{\phi}(1), \boldsymbol{\phi}(2), \ldots, \boldsymbol{\phi}(\nu)$. Partition $\boldsymbol{\Phi}$ and $\boldsymbol{k}^\top$ as follows:

$$\boldsymbol{\Phi} = \begin{bmatrix} \hat{\boldsymbol{\Phi}} & \tilde{\boldsymbol{\Phi}} \end{bmatrix}, \qquad \boldsymbol{k}^\top = \begin{bmatrix} \hat{\boldsymbol{k}}^\top & \tilde{\boldsymbol{k}}^\top \end{bmatrix},$$

where

$$\hat{\boldsymbol{\Phi}} = \begin{bmatrix} \boldsymbol{\phi}(1) & \boldsymbol{\phi}(2) & \ldots & \boldsymbol{\phi}(\nu) \end{bmatrix}, \qquad \hat{\boldsymbol{k}}^\top = \begin{bmatrix} k(1) & k(2) & \ldots & k(\nu) \end{bmatrix}$$

and $\tilde{\boldsymbol{\Phi}}$ and $\tilde{\boldsymbol{k}}^\top$ account for the remaining entries. The system (10) can be written as

$$\hat{\boldsymbol{\Phi}}\hat{\boldsymbol{k}} = -\tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{k}}. \tag{11}$$

If the columns $\boldsymbol{\phi}(i)$ are linearly independent, then

$$\hat{\boldsymbol{k}} = -\hat{\boldsymbol{\Phi}}^{-1}\tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{k}}. \tag{12}$$

Using this expression, we determine $\hat{\boldsymbol{k}}$ for arbitrary $\tilde{\boldsymbol{k}}$.

The following theorem provides a criterion for the linear independence of ν columns of $\mathbf{\Phi}$.

Theorem 2. *Let $\mathcal{A}$ be a subset of $\mathcal{E}$ of cardinality ν. The columns of $\mathbf{\Phi}$ that correspond to the elements of $\mathcal{A}$ are linearly independent if and only if the edges in $\mathcal{E} \setminus \mathcal{A}$ form a spanning tree of Γ.*

Proof. Without loss of generality, consider the first ν columns of $\mathbf{\Phi}$. Suppose that the edges that correspond to the columns of $\tilde{\mathbf{\Phi}}$ form a spanning tree of Γ. If we are given the ratios of the transition rates for the edges in $\mathcal{F}$, then, by using Condition 1 for the fundamental cycles, we can find the corresponding ratios for the edges in $\mathcal{E} \setminus \mathcal{F}$. In other words, the system (11) has a unique solution, $\hat{\boldsymbol{k}}$, for every $\tilde{\boldsymbol{k}}$. As is well known, a system of ν linear algebraic equations with ν unknowns has a unique solution if and only if the matrix of coefficients of the system is nonsingular. Hence, the columns of $\hat{\mathbf{\Phi}}$ are linearly independent.

Suppose now that the columns of $\hat{\mathbf{\Phi}}$ are linearly independent, and the edges that correspond to the columns of $\tilde{\mathbf{\Phi}}$ do not form a spanning tree of Γ. Then these edges form a graph, Θ, with $m - \nu = n - 1$ edges and $n' \leq n$ vertices. A tree can be defined as a connected graph with u vertices and $u - 1$ edges (see [4, Chapter 7]). If Θ were a tree, it would be one spanning all vertices of Γ, i.e. its spanning tree, which contradicts our assumption. Suppose that Θ contains $C > 1$ connected components each of which is a tree. Let these trees have $n_1, \ldots, n_C$ vertices, $\sum_{i=1}^{C} n_i = n'$. In this case the number of edges in Θ should be equal to

$$\sum_{i=1}^{C}(n_i - 1) = \sum_{i=1}^{C} n_i - C = n' - C.$$

This is less than the actual number $n - 1$; thus, we have come to a contradiction. Hence, at least one connected component of Θ is not a tree (this is also true if the connected component is unique). A tree is a connected graph with no cycles; therefore, Θ necessarily contains a cycle.

We now choose the transition rates for $\tilde{\boldsymbol{k}}$ in such a way that Condition 1 for this cycle is not satisfied. But since the columns of $\hat{\mathbf{\Phi}}$ are linearly independent, for this choice of $\tilde{\boldsymbol{k}}$ the system (11) has a solution, $\hat{\boldsymbol{k}}$, and the chain $X(t)$ with the corresponding transition rates is reversible by Theorem 1. Therefore, Condition 1 must be satisfied for all cycles in Γ (see Remark 1). We have come to a contradiction, which completes the proof.

Theorem 2 in fact states that the only consistent way to construct a reversible Markov chain is to find a spanning tree and use Condition 1 for the fundamental cycles. But it also gives more—it allows us to find (to guess) another spanning tree if we have already found one. This may be useful in certain situations, for example if we want to replace some edges in Δ. And if we have found another spanning tree, we do not need to determine the fundamental cycles in order to construct a reversible Markov chain; all we have to do is use the corresponding analogue of (12).

The fundamental cycle condition can be used to construct reversible Markov chains from reversible Markov chains by changing some of the transition rates. Suppose that $X(t)$ is reversible, and we change the ratio of the transition rates for some edge $(i, j) \in \mathcal{E}$. What needs to be done in order to preserve the property of reversibility? If we know the matrix $\mathbf{\Phi}$, we can then identify the fundamental cycles to which the edge (i, j) belongs, and change the transition rates for the corresponding edges in $\mathcal{E} \setminus \mathcal{F}$ in order for Condition 1 to be satisfied for each of these cycles. The resulting Markov chain will be reversible by Theorem 1. If the fundamental cycle to which the edge (i, j) belongs consists of edges that belong to no other cycle, then, in order for Condition 1 to be satisfied, we may change the transition rates for any other edge in this cycle. If the edge (i, j) does not belong to any fundamental cycle (the corresponding column of $\mathbf{\Phi}$ is a zero vector), then changing the transition rates for this edge will not affect reversibility.

Our spanning tree method can also be used to construct reversible Markov chains with a given stationary distribution. If we have a probability distribution $\boldsymbol{\sigma}$, then we give values to the transition rates for the edges in the spanning tree so that the detailed balance conditions are satisfied for $\boldsymbol{\sigma}$. Then, using Condition 1, we give values to the transition rates for the edges which are not in the spanning tree. The resulting Markov chain is reversible, and its stationary distribution is $\boldsymbol{\sigma}$.

4. Finding a spanning tree

Christofides [4, Chapter 7] described an algorithm that generates all spanning trees of a nondirected graph (for other algorithms, see references therein). So, if we do not want *all* spanning trees, finding *some* spanning tree of a graph in not a problem (see also [8, Chapter 2]). But we might be interested in the special case where the spanning tree has a linear structure. As follows from (4)–(6), in this case, for the stationary distribution, we can obtain simple formulae, which are similar to the expressions (1) and (2) for a birth–death process. It is easily seen that finding such a spanning tree is equivalent to finding a Hamiltonian path, i.e. an elementary path spanning all vertices of Γ. This problem also has an algorithmic solution; one of the possible ways of solving it is considered below.

Consider the directed graph, $\vec{\Gamma}$, obtained from Γ by replacing each edge with a pair of arcs having opposite orientations. Now add a vertex to $\vec{\Gamma}$ and connect this vertex to every other vertex with a pair of arcs having opposite orientations. In the resulting graph, $\vec{\Lambda}$, it is possible to find a Hamiltonian cycle (a simple cycle spanning all the vertices) or prove that it does not exist by using one of the algorithms described in [4, Chapter 10]. The graph $\vec{\Gamma}$ contains a Hamiltonian path if and only if $\vec{\Lambda}$ contains a Hamiltonian cycle. If we have found a Hamiltonian cycle in $\vec{\Lambda}$, then, removing the added vertex together with the incident arcs, we get a Hamiltonian path in $\vec{\Gamma}$ and Γ. Also, $\vec{\Gamma}$ contains a Hamiltonian path if it contains a Hamiltonian cycle. It is worth noting that methods for generating random spanning trees and Hamiltonian cycles exist (see, for example, [7] and references cited therein).

If we are using the formulae (4)–(6) for the stationary distribution, we would probably want to find a spanning tree in which the paths from the vertex r to other vertices are as short as possible. Using Dijkstra's algorithm [8, Chapter 3], it is possible to find the shortest paths from r to all other vertices in $\vec{\Gamma}$. The edges in the corresponding paths in Γ form a spanning tree of Γ. It is also possible to choose r to be a centre of Γ, i.e. a vertex such that the distance from this vertex to the farthest vertex is minimal [8, Chapter 8].

Sometimes we may need to establish that some set of edges forms a spanning tree of Γ. As stated above, if we have already found a spanning tree, then we can use Theorem 2. If we have not, we can use the following criterion [10, Corollary 2 to Theorem 3.3.1]: $n-1$ columns of the incidence matrix of Γ are linearly independent (using modulo 2 arithmetic) if and only if the corresponding edges form a spanning tree of Γ. It is clear that, if $n \gg \nu$, then it may be much more efficient to use Theorem 2. One important feature of Theorem 2 is that it is formulated in terms of the usual arithmetic operations, which allows us to use standard linear algebra routines.

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