Connection between the rate of convergence to stationarity and stability under perturbations for stochastic and deterministic systems

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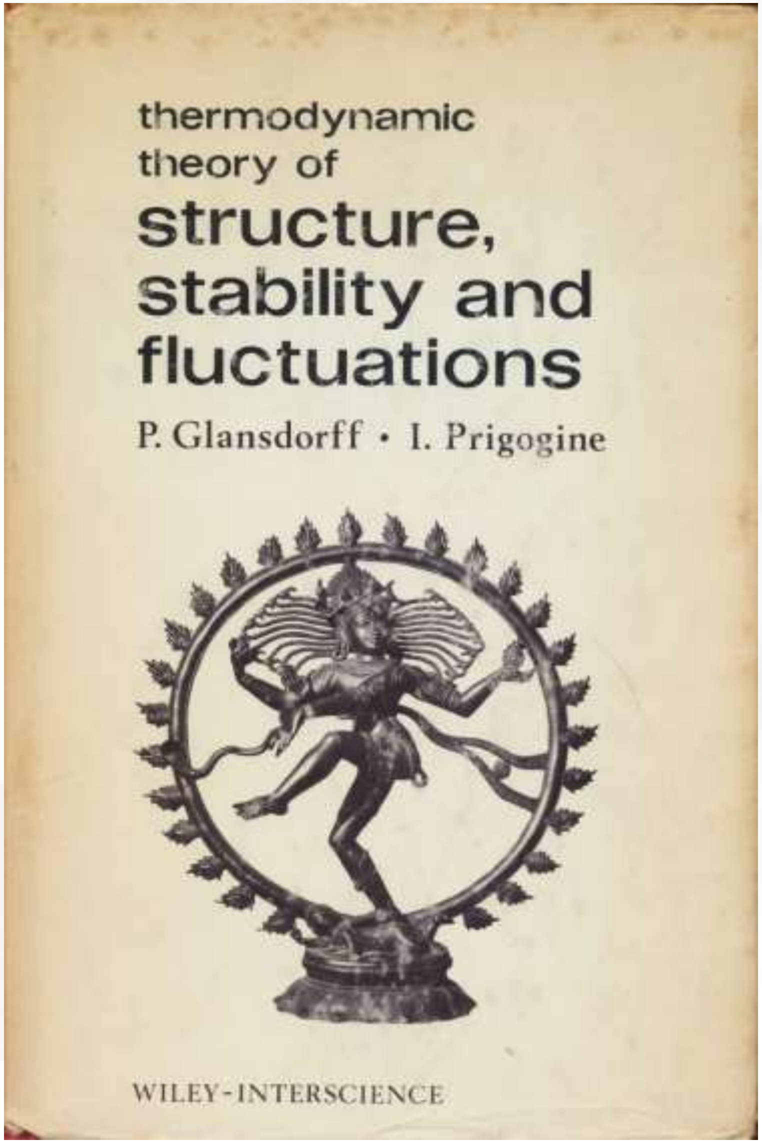
Dynamics Days Europe
Loughborough, England
4 September 2018

Ilya Prigogine (Nobel Prize in Chemistry, 1977)

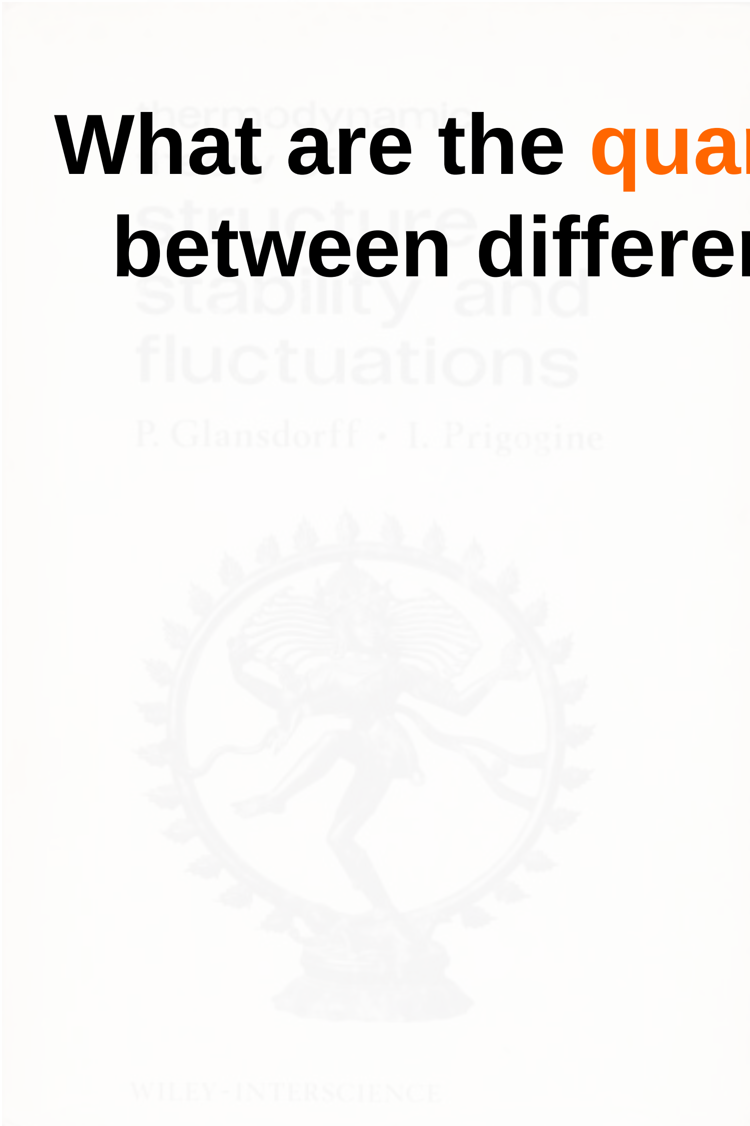


- Thermodynamic stability
- Hydrodynamic stability
- Chemical stability
- Lyapunov stability
- Asymptotic stability
- Structural stability
- Neutral stability
- Orbital stability

...

Ilya Prigogine (Nobel Prize in Chemistry, 1977)

What are the quantitative relationships between different types of stability?



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What are the quantitative relationships between different types of stability?

Our focus

Exponential Lyapunov-type stability → Stability under perturbations in the governing equation

perturbed initial conditions

perturbed "right-hand side"

Ergodicity!

…

What are the quantitative relationships between different types of stability?

Starting point: Markov chains!

Approach: perturbation bounds

Recent applications of the theory

Stat Comput (2016) 26:29–47
DOI 10.1007/s11222-014-9521-x

CrossMark

**Noisy Monte Carlo:
with approximate tr**

P. Alquier · N. Friel · R. Everit

Received

Abstract Monte Carlo algorithm
distribution π by simulating a Ma
kernel P such that π is invariant
are many situations for which it i
ble to draw from the transition ke
is the case with massive datasets,
expensive to calculate the likelih
for intractable likelihood models a
Gibbs random fields, such as those

Bernoulli **24**(4A), 2018, 2610–
https://doi.org/10.3150/17-BEJ

Perturbation th
via Wassersteir

DANIEL RUDOLF[1] and N

Perturbation theory for Markov c
probabilities of Markov chains ar
ful and flexible bounds on the dis
them satisfies a Wasserstein ergod
mate Markov chain Monte Carlo (
based on Lyapunov functions, we
assumptions. In an autoregressive
ory by showing quantitative estim
Metropolis–Hastings and stochasti

Keywords: big data; Markov chain

Statistics and Computing
https://doi.org/10.1007/s11222-018-9817-3

**Informed sub-samplin
datasets**

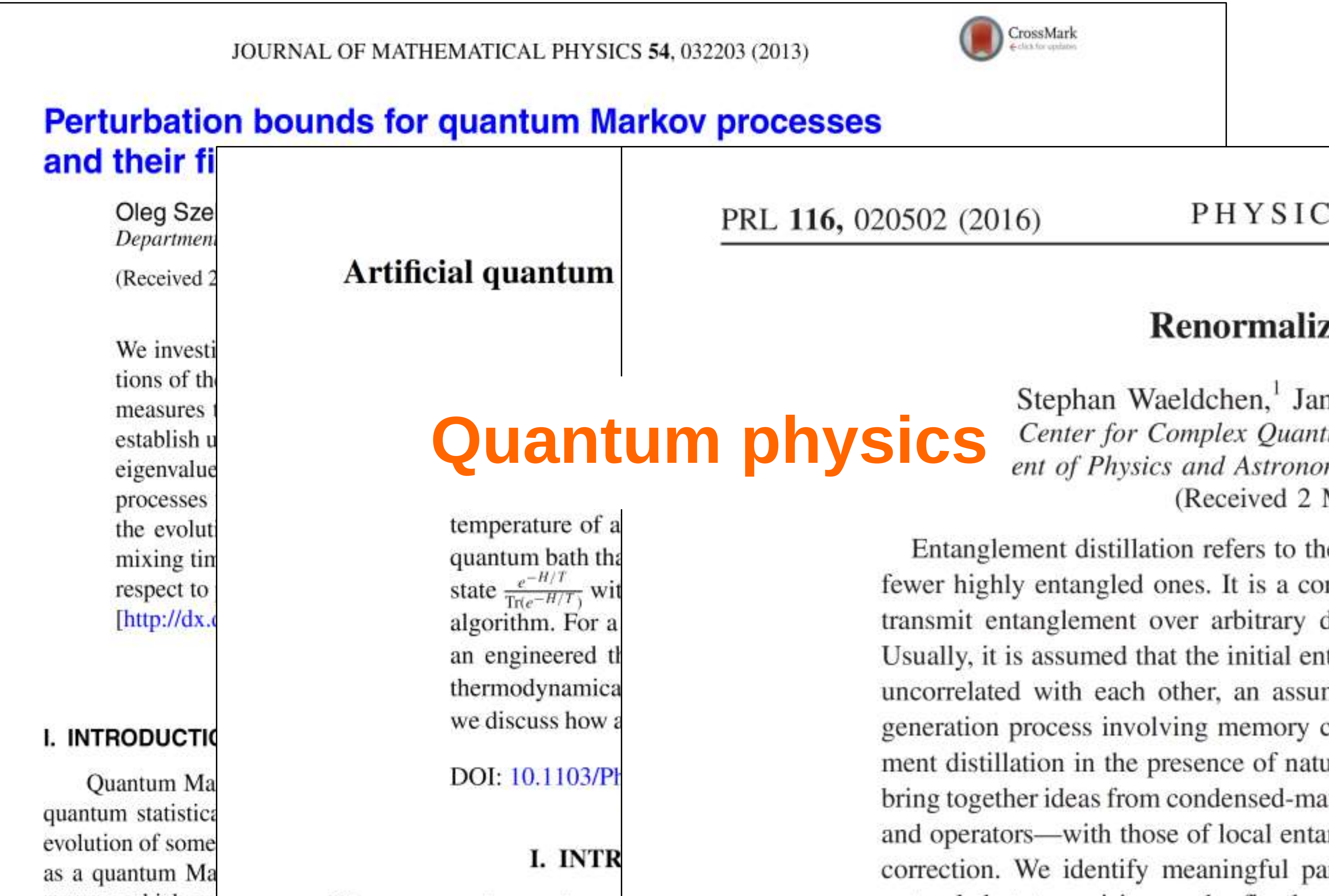
Florian Maire[1,2] · Nial Friel[1,2] · Pi

Received: 26 June 2017 / Accepted: 4 June 2

Abstract
This paper introduces a framework

Statistics and algorithms

Recent applications of the theory



Quantum physics

Recent applications of the theory

J Stat Phys (2016) 162:312–333
DOI 10.1007/s10955-015-1409-4

Response Operators for Markov P
State Space: Radius of Convergen
Response Theory for Axiom A Sys

Valerio Lucarini[1,2]

Received: 8 July 2015 / Accepted: 24 October 2015 / Publis

Abstract Using straightforward linear algebra we
impact of small perturbations to finite state Ma
for studying empirically constructed—e.g. from
of model simulations—finite state approximation
results concerning the convergence of the statistica

Rough parameter depende
the role of Ruelle-Pollicott

Mickaël David Chekroun, J. David Neelin, Dmitri Kondrasho

Department of Atmospheric and Oceanic Sciences and Institute of Geophysi

Contributed by James C. McWilliams, November 22, 2013 (sent for review Au

Despite the importance of uncertainties encountered in climate
model simulations, the fundamental mechanisms at the origin
g-term model statistics remain unclear.
ws in the atmosphere and oceans ex-
patterns. These patterns, while evolv-
ifest characteristic frequencies across a
from intraseasonal through interdeca-
dal. Based on modern spectral theory of chaotic and dissipative
dynamical systems, the associated low-frequency variability may
be formulated in terms of Ruelle-Pollicott (RP) resonances. RP res-
onances encode information on the nonlinear dynamics of the
system, and an approach for estimating them—as filtered through
an observable of the system—is proposed. This approach relies on
an appropriate Markov representation of the dynamics associated
with a given observable. It is shown that, within this representa-
tion, the spectral gap—defined as the distance between the sub-
dominant RP resonance and the unit circle—plays a major role in
the roughness of parameter dependences. The model statistics are
the most sensitive for the smallest spectral gaps; such small gaps
turn out to correspond to regimes where the low-frequency vari-

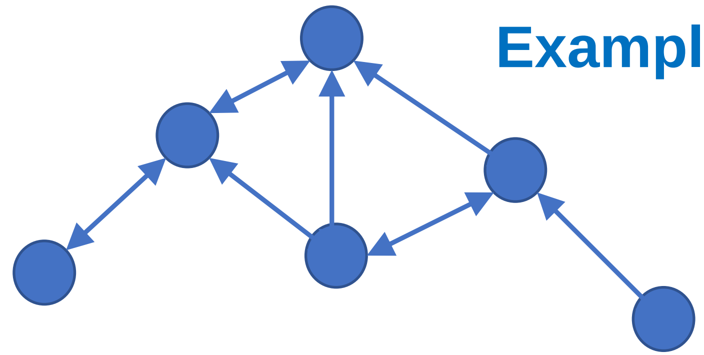
Climate science

Markov chains (finite, continuous-time for now)

Motivation

- Logic of development for this approach
- Convenient illustration of the main concepts
- Practical significance: stochastic models (e.g., **chemical master equation**)

Example



$$\mathbf{Q} = (q_{ij})$$

transition rate matrix (generator)

$$\mathbf{p}(t) = (p_i(t))$$

state probability vector

$$d\mathbf{p}(t)/dt = \mathbf{p}(t)\mathbf{Q}$$

governing equation (Kolmogorov equation)

A classic result and its improvement

Let $\|\cdot\|$ be the 1-norm (**total variation** distance). Suppose that

$$\left\|\mathbf{p}^{(1)}(t)-\mathbf{p}^{(2)}(t)\right\| \le Ce^{-bt}\left\|\mathbf{p}^{(1)}(0)-\mathbf{p}^{(2)}(0)\right\|$$

Ergodicity!

Consider perturbed chain with $\tilde{\mathbf{Q}}, \ \tilde{\mathbf{p}}(t); \quad \mathbf{E}=\tilde{\mathbf{Q}}-\mathbf{Q}, \quad \mathbf{z}(t)=\tilde{\mathbf{p}}(t)-\mathbf{p}(t)$

Assume that $\mathbf{z}(0)=\mathbf{0}$ (for now)

Perturbation bound:

$$\sup_{t\ge 0}\|\mathbf{z}(t)\| \le 12b^{-1}C\,\|\mathbf{E}\|$$

(Zeifman and Isaacson, *Stoch Proc Appl* 1994)

Improved bound:

$$\sup_{t\ge 0}\|\mathbf{z}(t)\| \le b^{-1}(\underline{\log C}+1)\|\mathbf{E}\|$$

(Mitrophanov, *J Appl Probab* 2003)

A classic result and its improvement

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Improved bound:

$$\sup_{t\ge 0}\|\mathbf{z}(t)\| \le b^{-1}(\underline{\log C}+1)\|\mathbf{E}\|+\|\mathbf{z}(0)\|$$

(if nonzero)

(Mitrophanov, *J Appl Probab* 2003)

Features of the improved perturbation bound

Convergence rate is the **main** determinant of stability under perturbations (because of the log)

Works for **both** transient *and* equilibrium regimes

Handles **combined** perturbations: initial conditions *and* the “right-hand side”

Extends to the **time-inhomogeneous, countable** state-space case (Zeifman et al., *Inform Primen 2011*; *Stoch Models* 2012; *Mathematics* 2018)

Related perturbation results

$\{0,1,\ldots,N\}$ – state space

$\lambda_0 = 0, \lambda_1, \ldots, \lambda_N$ – eigenvalues of the generator $\mathbf{Q}$; $\rho = \min_{i>0} |\lambda_i|$

Ergodicity!

$\boldsymbol{\pi} = (\pi_i)$ – unique steady state

Then

$$\|\tilde{\boldsymbol{\pi}} - \boldsymbol{\pi}\| \le \underline{N} \rho^{-1} \|\mathbf{E}\|$$

(Mitrophanov, *Theory Probab Appl* 2006)

Suppose now that $\mathbf{Q}$ is **reversible** ("detailed balance"):

$$\pi_i q_{ij} = \pi_j q_{ji}$$

Then

$$\sup_{t\ge 0} \|\mathbf{z}(t)\| \le \|\mathbf{z}(0)\| + \underline{66e(e-1)^{-1} N} \rho^{-1} \|\mathbf{E}\|$$

(Mitrophanov, *J Appl Probab* 2004)

Now, what about discrete time?

General state space $(S, \Im)$; P and $\tilde{P}$ – unperturbed and perturbed transition operators

Unique invariant measure: $\pi P = \pi$

Uniform ergodicity: $\left\| p_0 P^n - \pi \right\| \le C \rho^n$ for all $p_0, \; n \in \mathbb{Z}_+$

Denote $p_n = p_0 P^n, \quad \tilde{p}_n = \tilde{p}_0 \tilde{P}^n, \quad z_n = \tilde{p}_n - p_n, \quad E = \tilde{P} - P$

Perturbation bound:

$$\sup_{n \in \mathbb{N}} \left\| z_n \right\| \le \left\| z_0 \right\| + \left(\hat{n} + \frac{C \rho^{\hat{n}}}{1 - \rho} \right) \left\| E \right\|,$$

where $\hat{n} = \left\lceil \log_\rho C^{-1} \right\rceil$

(Mitrophanov, *J Appl Probab* 2005)

Now, what about discrete time?

…But, what if the unperturbed chain is **not** uniformly ergodic?

Geometric ergodicity: $\left\| p_0 P^n - \pi \right\| \le C(p_0)\rho^n, \quad n \in \mathbb{Z}_+$

This is a subject of much ongoing work ("BIG DATA"!)

One approach is to extend the results above using the **Wasserstein** distance (instead of total variation): Rudolf and Schweizer, *Bernoulli* 2018

$$W(\mu,\nu) = \inf_{\gamma \in M(\mu,\nu)} \int_G m(x,y)\, d\gamma(x,y)$$

[for a metric $m(x,y)$]

Now, what about nonlinear dynamical systems?

$d\mathbf{x}/dt = \mathbf{f}(\mathbf{x}, t)$ − unperturbed system (finite-dimensional)

$d\tilde{\mathbf{x}}/dt = \mathbf{f}(\tilde{\mathbf{x}}, t) + \mathbf{d}(\tilde{\mathbf{x}}, t)$ − perturbed system; $\|\mathbf{d}(\tilde{\mathbf{x}}, t)\| \le d$

Assumption − the unperturbed system is *globally contracting*:

$$\left\|\mathbf{x}^{(1)}(t) - \mathbf{x}^{(2)}(t)\right\| \le e^{-bt}\left\|\mathbf{x}^{(1)}(0) - \mathbf{x}^{(2)}(0)\right\|$$

"Ergodicity"!

For $\mathbf{z}(t) = \tilde{\mathbf{x}}(t) - \mathbf{x}(t)$,

$$\sup_{t\ge 0}\|\mathbf{z}(t)\| \le \|\mathbf{z}(0)\| + b^{-1}d$$

(Del Vecchio and Slotine, *IEEE Transact Automatic Control* 2013; Gyorgy and Del Vecchio, *PLOS Comput Biol* 2014)

Open problem: stochastic perturbations

$d\mathbf{x}/dt = \mathbf{f}(\mathbf{x}, t)$ – unperturbed system (deterministic)

$d\tilde{\mathbf{x}}/dt = \mathbf{f}(\tilde{\mathbf{x}}, t) + Noise$ – perturbed system (stochastic)

Assumption – the unperturbed system is *globally contracting*:

$$\left\| \mathbf{x}^{(1)}(t) - \mathbf{x}^{(2)}(t) \right\| \le e^{-bt} \left\| \mathbf{x}^{(1)}(0) - \mathbf{x}^{(2)}(0) \right\|$$

"Ergodicity"!

Then, does the following hold?

$$\mathrm{Dist}(\tilde{\mathbf{x}}(t), \mathbf{x}(t)) \le \mathrm{Dist}(\tilde{\mathbf{x}}(0), \mathbf{x}(0)) + \mathrm{Const} \times b^{-1} \mathrm{Magnitude}(Noise)$$

?

BOTTOM LINE

Exponential Lyapunov-type stability → Stability under perturbations in the governing equation

perturbed initial conditions

perturbed “right-hand side”

- Ergodicity bounds can have an *extra benefit*: a perturbation bound

- **This may be a *universal* phenomenon** (think *central limit theorem*, *large numbers law*, etc.)

Stochastic processes to investigate for this

- Markov jump processes (general state space, continuous time)
- Diffusions (recent progress)
- Stochastic (partial) differential equations
- Semi-Markov processes
- Markov processes in a random environment
- Branching processes
- . . .

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